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ON A PROBABILISTIC STABILITY THEORY FOR IMPERFECTION SENSITIVE STRUCTURES

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Abstract—The concept of almost sure sample stability and sample stability in probability are formulated for elastic systems. Using a Koiter type approach these concepts are used in the analysis of imperfection sensitive structures. The applied load and the initial geometric imperfections are introduced into the analysis as random quantities. A compressed beam of finite length on a nonlinear elastic foundation is used in an example calculation.

INTRODUCTION

Probabilistic methods are being used extensively in the modelling of many engineering problems as it becomes apparent that this approach may yield a physically more meaningful result. In the case of imperfection sensitive structures this is particularly true since both the applied load and the initial imperfection are random quantities.

The existing literature dealing with the probabilistic features of imperfection sensitive systems can be divided into two basic groups. The first group includes the work by Bolotin[1], Thompson[2] and Roorda[3]. The characteristic of this group is the use of deterministic critical load—initial imperfection relationships as transfer functions. Thus, given the probabilistic description of the initial imperfection, the probabilistic description of the critical load may be obtained. A restriction that results from this approach is that the initial imperfection must be of a certain shape with a random amplitude, or a summed series of shapes with random amplitudes. In order to overcome this restriction and the difficulties which are associated with the above approach, a second group of papers, published by Amazigo[4], Fraser and Budiansky[5], Amazigo *et al.*[6], van Slooten and Soong[7] and Fersht[8], represent the initial imperfection as a random process. Then, making use of the methods of stochastic differential equations, they obtain a relationship between the critical load and the initial imperfection. The imposition of the assumption that the random process is ergodic yields a deterministic critical load which, in the asymptotic case, depends only on the spectral density of the imperfection.

The result that the critical load is deterministic is at first a surprise, as intuitively it seems that a random imperfection should yield a random critical load. This paradox is, however, due to the ergodicity assumption. That is, the critical load depends on certain integrals or spatial averages of the initial imperfection and once the ergodicity assumption is made these

averages automatically become equivalent to statistical moments. Thus, the critical load is a deterministic quantity which depends only on these averages. In essence this approach does not rely on defining the initial imperfection as a random variable; rather it makes use of a different definition of a deterministic imperfection. That is, the imperfection is specified by certain spatial averages, in this case the spectral density.

In both approaches to the problem the stability definitions used are either not well defined or the implications of these definitions are not made clear. Thus, the first objective of the present paper is the development of well defined stability criteria. These criteria will then be combined with a Koiter style analysis to yield a general analysis for randomly imperfect systems in which the applied load is a random variable and initial imperfection is represented by a random process.

STABILITY CRITERIA

Probabilistic stability criteria may be derived from several different points of view. In particular the concepts of almost sure stability, stability in probability, and moment stability are in common use. The application of any of these definitions depends to a great extent on the problem under consideration, the results desired, as well as the ease or possibility of using a given definition. In applications to real systems, it has been suggested by Kosin[9] that almost sure sample stability properties are desirable. The justification of this point of view is due to two different features; firstly, almost sure sample stability properties yield a close analogy with the corresponding deterministic stability properties and, secondly, samples (not averages) represent actual events or observations. The significance of almost sure sample stability is that it guarantees that almost every sample will satisfy the stability requirement or, stated otherwise, every member of the ensemble except some with zero probability will satisfy the stability requirement.

In the following discussion it is assumed that the initial imperfection is a smooth process. A smooth process is one which with probability one possesses continuous sample derivatives. This point of view leads to the concept of the *random energy functional*, which is a functional of a random displacement, a random load and the random initial imperfection, all of which are not independent. This functional can be viewed as a collection of deterministic energy functionals upon which has been induced a probability measure by way of the initial imperfection. This concept will lead to an almost sure sample stability criterion.

It is further assumed that any given load level for any member of the ensemble is attained through the addition of arbitrarily small increments of load, all of the same sign. As a preliminary to the discussion of the probabilistic stability criterion it is necessary to recall some of the features of the deterministic stability criterion.

In the deterministic theory, an equilibrium position is defined to be a stable one when the equilibrium configuration corresponds to a local minimum of the potential energy. An unstable equilibrium configuration is defined to occur when it no longer represents a local minimum. The necessary conditions for the stability of a conservative static system are

$$\delta V = 0, \tag{1a}$$

$$\delta^2 V \geq 0, \tag{1b}$$

and if in the last equation the sign of equality holds,

$$\delta^3 V = 0, \tag{2a}$$

$$\delta^4 V \geq 0, \tag{2b}$$

and if in the last equation the sign of equality holds,

$$\delta^5 V = 0, \tag{3a}$$

$$\delta^6 V \geq 0, \tag{3b}$$

and so on, where V is the potential energy and $\delta^n V$ represents the nth variation of the potential energy. Equation (1a) represents the requirement that the system under consideration be in a state of equilibrium and the remaining equations yield the necessary conditions for a proper local minimum of the potential energy.

From a purely mathematical point of view the above definition is entirely adequate; however, when the stability of a real system is investigated certain other features must be considered. Firstly, a real system cannot occupy unstable equilibrium states but will deform until a stable state is found. Secondly, the stable states occupied after such a deformation cannot be considered to be of practical usefulness since the system has passed through a region of instability. Thus, it seems that a more practical and at the same time a more precise stability definition would be the following: the system is stable when

$$P_{cr} > P, \tag{4}$$

where P_{cr} represents the critical load and P represents the applied load. Here P_{cr} is defined to be the smallest load which places the system in an unstable critical state of equilibrium. The requirement that the critical state be unstable is equivalent to assuming that a critical load does in fact exist. That is, when equation (1a) holds, the equality in equation (1b) holds and any of the remaining equations are violated. It is noted in passing that P_{cr} is in no way related to the applied load, rather it is a characteristic of the system under consideration.

Consider now the case of a *restricted random potential energy functional* which is defined to be that combination of a random load, a random deflection and the random initial imperfection such that almost every sample occupies its first unstable critical state of equilibrium. If this is not possible for a given sample, this sample is rejected. The random load so defined is the critical load for the ensemble and the probabilistic measure associated with this load is defined by some transformation of the probabilistic measure of the initial imperfection. Thus, almost sure sample stability of the ensemble occurs when

$$\text{Prob}\{P_{cr} > P\} = 1 \tag{5}$$

where P_{cr} represents the random critical load and P represents the random applied load. If this equation is satisfied the implication is that the equilibrium configurations of almost every sample are uniquely defined for all loads less than or equal to the applied load. As can easily be appreciated this result is very powerful, and unfortunately, in some cases may be too restrictive. For example, if the applied load or the initial imperfection has a probability measure defined on the extended real axis, then it is impossible to satisfy equation (5). In these cases it becomes necessary to consider a criterion which is based on a measure of stability, or reliability. Thus, the second stability definition is that of sample stability in probability or more simply, reliability. That is

$$R = \text{Prob}\{P_{cr} > P\} \tag{6}$$

where R is the reliability.

APPLICATION OF THE STABILITY CRITERION

The method of analysis presented by Koiter[10] is a very systematic and concise procedure by which deterministic imperfection sensitive structures may be treated. In the probabilistic case this method can be applied with equal ease.

It is convenient to hypothesise that the initial random imperfection represents the difference between the actual samples, or members of the ensemble, and an idealized model. This model represents the "perfect" form and is completely deterministic. As has been previously noted, the point of view is taken that the *restricted load*, which yields the *restricted energy functional* is the lowest random load which will place every sample in an unstable critical state of equilibrium. Also, if it is not possible to place a given sample in such a state, this sample is rejected. The random load so obtained is defined to be the critical load of the ensemble. Based on the assumption imposed on the initial imperfection, the method employed by Koiter is valid and may be applied to the restricted energy functional in order to obtain a relationship between the ensemble critical load and the initial imperfection. Thus it is sufficient to present the final restricted energy expression, that is,

$$F(a) = B_1 a + (\bar{\lambda}_c - \lambda_1) A_2' a^2 + A_n a^n. \tag{7}$$

This expression contains only the predominant terms from the energy and is thus a first approximation. The variable 'a' can be considered to be a deflection parameter and is the only degree of freedom in the system, λ_1 is the critical load of the model, $\bar{\lambda}_c$ is the critical load of the imperfect system and is the quantity which is being evaluated. The terms B_1, A_2', A_n are functionals which are characteristic of the system under consideration. The terms A_2', A_n contain no influence from the initial imperfection and can be derived from the model of the system. The term B_1 is linear in the imperfection. Only the linear term in the imperfection is retained since, on account of the smallness of the imperfection, it will have a predominant influence.

For equilibrium of the restricted system it is necessary that the first variation of the energy be zero; therefore

$$\frac{\mathrm{d}F(a)}{\mathrm{d}a} = B_1 + 2(\bar{\lambda}_c - \lambda_1) A_2' a + n A_n a^{n-1} = 0. \tag{8}$$

Since the system is in a critical state of equilibrium the second variation is also equal to zero, hence

$$\frac{\mathrm{d}^2 F(a)}{\mathrm{d}a^2} = 2(\bar{\lambda}_c - \lambda_1) A_2' + n(n-1) A_n a^{n-2} = 0. \tag{9}$$

Also, the definition of the critical load requires that the critical state be unstable; thus, for $n > 2$,

$$\begin{aligned} \frac{\mathrm{d}^n F(a)}{\mathrm{d}a^2} &= n!\, A_n \neq 0, \qquad n = \text{odd} \\ &< 0, \qquad n = \text{even}. \end{aligned} \tag{10}$$

The solution of equations (8) and (9) for $\bar{\lambda}_c$ yields the required relationship for the critical load in terms of the initial imperfection and various parameters of the system under consideration. The form of equations (9) and (10) also yield further information of interest. Equation (9) represents the "stability boundary" for the imperfect system and it contains no influence from the initial imperfection. Thus the stability boundary is given by a deterministic relationship which may be obtained from an analysis of the model. Equation (10) determines the stability or instability of the critical state; this quantity is completely deterministic and can also be evaluated from an analysis of the model. This result is extremely useful because it implies that the existence of the random critical load depends only on certain characteristics of the model. A deterministic analysis of the model should therefore always precede the probabilistic analysis of the real structure. In fact, the analysis of the model determines whether or not the real structure is imperfection sensitive.

The two classes of structures which have been discussed at great length in the existing "deterministic" literature on imperfection sensitivity are the so-called asymmetric ($n = 3$) and symmetric ($n = 4$) cases. Many of the commonly used structural forms fall within one of these two categories and will therefore be dealt with in some detail. The methods used, however, could equally well be applied for higher values of n.

(1) *Asymmetric case* ($n = 3$, $A_3 \neq 0$)

For the asymmetric case the equilibrium equation is

$$B_1 + 2(\bar{\lambda}_c - \lambda_1)A_2'a + 3A_3 a^2 = 0, \tag{11}$$

and the stability boundary is given by

$$2(\bar{\lambda}_c - \lambda_1)A_2' + 6A_3 a = 0. \tag{12}$$

Solving these equations for the lowest value of the ensemble critical load yields

$$\frac{\bar{\lambda}_c}{\lambda_1} = 1 - \left[\frac{3A_3 B_1}{A_2'^2 \lambda_1^2}\right]^{1/2}. \tag{13}$$

This expression can be simplified by setting $\lambda_c = \bar{\lambda}_c/\lambda_1$ and $y_A = (3A_3 B_1)/(A_2'^2 \lambda_1^2)$, which reduces equation (13) to

$$\lambda_c = 1 - y_A^{1/2}. \tag{14}$$

The quantity y_A will be called the *imperfection parameter*. It is a linear functional of the initial imperfection which has been weighted by certain constants that are characteristics of the system under consideration. It should be noted that y_A is defined for positive values only because the system does not buckle if y_A is negative. Thus those samples which contain imperfections that lead to negative values of y_A are rejected.

For the investigation of stability it is convenient to define the variable

$$\lambda_R = \lambda_c - \lambda, \tag{15}$$

where λ is the random applied load divided by λ_1. The quantity λ_R may be thought of as the random reserve load. Substitution for λ_c in equation (15) from equation (14) yields

$$\lambda_R = 1 - \lambda - y_A^{1/2}. \tag{16}$$

In order that an expression for the reliability can be obtained it becomes necessary to assume that the joint density function of λ and y_A, designated as $f_{\lambda y_A}(\lambda, y_A)$, can be obtained. The implications of this assumption will be discussed in a later section.

Use of this density function allows computation of the joint density function of λ_R and y_A. The latter can be written as

$$f_{\lambda_R y_A}(\lambda_R, y_A) = f_{\lambda y_A}(1 - \lambda_R - y_A^{1/2}, y_A). \tag{17}$$

Thus the reliability, as defined in equation (6), is

$$R = \text{Prob}(\lambda_R > 0)$$

$$= \int_{0^+}^{\infty} \left\{ \int_0^{\infty} [f_{\lambda y_A}(1 - \lambda_R - y_A^{1/2}, y_A)] \, dy_A \right\} d\lambda_R. \tag{18}$$

In the above integrals the lower limits deserve special attention. The 0^+ occurs because of the inequality on λ_R and the 0 because y_A is not defined for negative values.

(2) *Symmetric case* ($n = 4$, $A_4 < 0$)

The equilibrium equation for this case is given by

$$B_1 + 2(\bar{\lambda}_c - \lambda_1)A_2' a + 4A_4 a^3 = 0 \tag{19}$$

and the stability boundary by

$$2(\bar{\lambda}_c - \lambda_1)A_2' + 12A_4 a^2 = 0. \tag{20}$$

Solving these equations for the lowest value of the critical load yields

$$\frac{\bar{\lambda}_c}{\lambda_1} = 1 - \left[\left(\frac{27A_4}{8A_2'^3 \lambda_1^3} \right)^{1/2} B_1 \right]^{2/3}. \tag{21}$$

In order to simplify let $y_s = [(27A_4)/(8A_2'^3\lambda_1^3)]^{1/2} B_1$.
Using the previous definition for λ_c then yields

$$\lambda_c = 1 - y_s^{2/3}. \tag{22}$$

The quantity y_s will be called the imperfection parameter which, like y_A, is a linear functional of the initial imperfection.

Substitution for λ_c in equation (15) from equation (22) yields

$$\lambda_R = 1 - \lambda - y_s^{2/3}. \tag{23}$$

Assuming that the joint density function of λ and y_s, i.e. $f_{\lambda y_s}(\lambda, y_s)$, can be obtained it follows readily that the joint density function of λ_R and y_s is

$$f_{\lambda_R y_s}(\lambda_R, y_s) = f_{\lambda y_s}(1 - \lambda_R - y_s^{2/3}, y_s). \tag{24}$$

The reliability for the symmetric case, using equation (6) is therefore

$$R = \text{Prob}(\lambda_R > 0) = \int_{0^+}^{\infty} \left\{ \int_{-\infty}^{\infty} [f_{\lambda y_s}(1 - \lambda_R - y_s^{2/3}, y_s] \, dy_s \right\} d\lambda_R . \tag{25}$$

In the above integrals it is noted that the range of integration for y_s extends from $-\infty$ to ∞ because λ_c is defined for all values of y_s.

GAUSSIAN INPUTS

Before continuing it is necessary to discuss the implications of equations (18) and (25). These expressions are completely general (within the assumptions of smallness imposed on the initial imperfection) and are valid for all possible joint density functions of λ and y_A (or y_s). Thus the immediate difficulty is the generation of $f_{\lambda y_A}(.,.)$ and $f_{\lambda y_S}(.,.)$.

It can be stated that the imperfection parameters y_A and y_s are random variables which are specified by a linear functional relationship with the initial imperfection. This functional, B_1, is in general a multiple integral which extends over the dimensions of the structure under consideration. The initial imperfection is a random process spanning the same dimensions. The utilization of integrals of this form for random processes is justified in the present context from the point of view of sample function integrability which is assured by the assumption that the initial imperfection is a smooth process.

The applied load, on the other hand, involves no spatial integral, being merely a random variable.

It follows, theoretically, that the input to the problem should be an infinite set of joint density functions of the applied load and the initial imperfection process. If such information is available, or can be obtained, then the immediate difficulty is the calculation of the joint density function of the applied load and the imperfection parameter.

For arbitrary joint density functions of the applied load and the initial imperfection it is, as a rule, impossible to obtain an analytic expression for the density function of the imperfection parameter. Exceptions to this rule occur; (i) when the applied load and the initial imperfection are jointly Gaussian, or (ii) when the applied load and the initial imperfection are independent with the initial imperfection being a Gaussian process. The simplification in these cases arises because of the linear relationship between the imperfection parameter and the initial imperfection process with the result that the marginal density function of the imperfection parameter is Gaussian.

The second case is of considerable interest in that it allows an arbitrary density function for the applied load. The independence of the applied load and the initial imperfection may be justified from the point of view that initial imperfections are a result of the manufacture of the structural system, whereas the load is applied only after the manufacturing process is complete and is independent of the manufacturing process. This argument, of course, precludes possible loading eccentricities which would induce a dependence between the applied load and the initial imperfection.

In the following work it is assumed; (i) that the applied load and the initial imperfection are independent, (ii) that the applied load is a Gaussian random variable, and (iii) that the initial imperfection is a Gaussian random process. Perry[11] has shown by an extensive process of accurate measurements that for certain simple structures the initial imperfections are distributed as Gaussian variables. It is therefore felt that the Gaussian assumption may yield physically reasonable results.

(1) *Asymmetric case*

On the basis of the above assumptions it follows that the imperfection parameter, y_A is completely specified by its mean

$$\bar{y}_A = \frac{3A_3}{(A_2')^2\lambda_1^2} E\{B_1\} \tag{26}$$

and its standard deviation

$$\sigma_A = \frac{3|A_3|}{(A_2')^2\lambda_1^2} [E\{B_1^2\} - E^2\{B_1\}]^{1/2}. \tag{27}$$

Thus the joint density function of applied load, λ, and y_A is

$$f_{\lambda y_A}(\lambda, y_A) = \frac{1}{\frac{1}{2}\left[1 - \operatorname{erf}\left(\frac{-\bar{y}_A}{\sigma_A\sqrt{2}}\right)\right]} \frac{1}{2\pi\sigma\,\sigma_A} \exp\left[\frac{-(\lambda - \bar{\lambda})^2}{2\sigma^2}\right] \exp\left[-\frac{(y_A - \bar{y}_A)^2}{2\sigma_A^2}\right] \tag{28}$$

where $\bar{\lambda}$ and σ are the mean and standard deviation, respectively, of the applied load. The constant $\{\frac{1}{2}[1 - \operatorname{erf}(-\bar{y}_A/\sigma_A\sqrt{2})]\}^{-1}$ is introduced as a normalization factor because no critical loads exist for negative value of y_A. It follows immediately from equation (18) that the reliability is

$$R = \frac{1}{\frac{1}{2}\left[1 - \operatorname{erf}\left(\frac{-\bar{y}_A}{\sigma_A\sqrt{2}}\right)\right] 2\pi\sigma\,\sigma_A} \int_{0+}^{\infty}\int_{0}^{\infty} \exp\left[-\frac{(y_A - \bar{y}_A)^2}{2\sigma_A^2}\right] \times \exp\left[-\frac{(\lambda_R - 1 + y_A^{1/2} + \bar{\lambda})^2}{2\sigma^2}\right] dy_A\, d\lambda_R. \tag{29}$$

Integrating with respect to λ_R yields the result

$$R = \frac{1}{2} - \frac{1}{\left[1 - \operatorname{erf}\left(\frac{-\bar{y}_A}{\sigma_A\sqrt{2}}\right)\right]\sqrt{2\pi}\,\sigma_A} \int_0^{\infty} \left\{\exp\left[-\frac{(y_A - \bar{y}_A)^2}{2\sigma_A^2}\right] \operatorname{erf}\left(\frac{-1 + y_A^{1/2} + \bar{\lambda}}{\sigma\sqrt{2}}\right)\right\} dy_A. \tag{30}$$

This result is valid for all systems whose imperfection sensitivity is represented by equation (14).

It should be noted that the parameter R in the asymmetric case represents only the reliability of the members of the ensemble which are capable of buckling. Thus in the cases considered, if the proportion of samples rejected becomes very large, the results obtained may be misleading. The reliability, against buckling, of the entire ensemble may readily be calculated from the relationship

$$R_T = \tfrac{1}{2}\left[1 + \operatorname{erf}\left(\frac{-\bar{y}_A}{\sigma_A\sqrt{2}}\right)\right] + \tfrac{1}{2}R\left[1 - \operatorname{erf}\left(\frac{-\bar{y}_A}{\sigma_A\sqrt{2}}\right)\right], \tag{31}$$

in which R_T is the reliability of the entire ensemble. A graphical insight into the relationship between reliability R_T, and the inputs $\bar{\lambda}$, σ, $\bar{y}_A$ and σ_A can be obtained from Figs. 1–3.

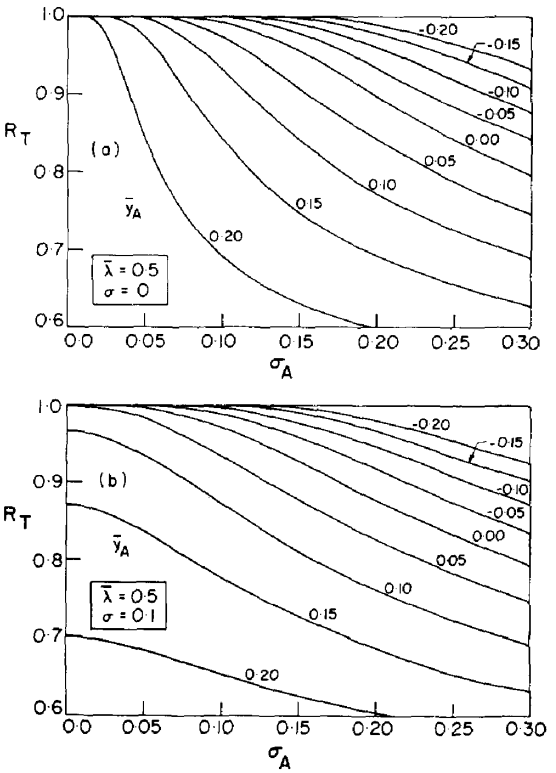

Fig. 1. Asymmetric system: Influence of the inputs $\bar{\lambda}$, σ, $\bar{y}_A$ and σ_A on R_T.

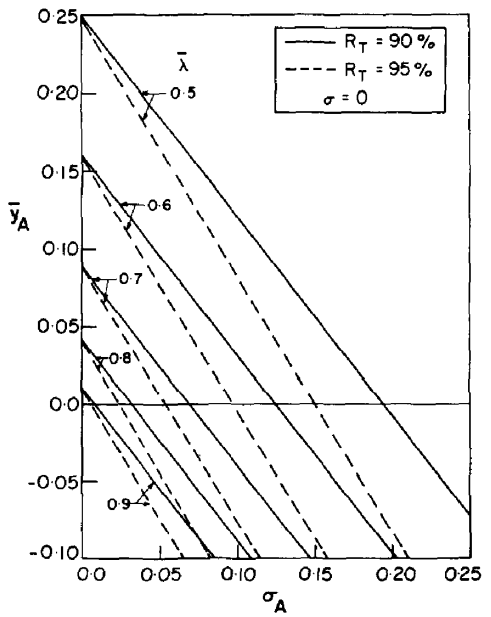

Fig. 2. Asymmetric system: Equal levels of R_T in the $\bar{y}_A - \sigma_A$ plane as $\bar{\lambda}$ varies.

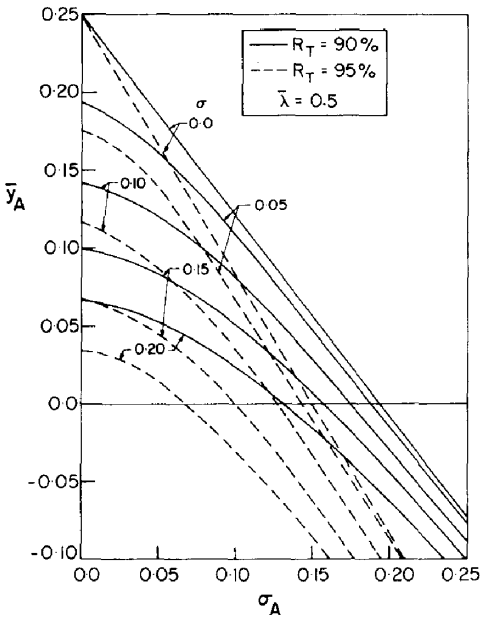

Fig. 3. Asymmetric system: Equal levels of R_T in the $\bar{y}_A - \sigma_A$ plane as σ varies.

(2) *Symmetric case*

In the symmetric case the mean and standard deviations of the imperfection parameter are

$$\bar{y}_s = \left(\frac{27A_4}{8(A_2')^3\lambda_1^3}\right)^{1/2} E\{B_1\}, \tag{32}$$

and

$$\sigma_s = \left(\frac{27A_4}{8(A_2')^3\lambda_1^3}\right)^{1/2} [E\{B_1^2\} - E^2\{B_1\}]^{1/2}, \tag{33}$$

respectively. The joint density function of λ and y_s is

$$f_{\lambda y_s}(\lambda, y_s) = \frac{1}{2\pi\sigma\,\sigma_s} \exp\left[-\frac{(\lambda - \bar{\lambda})^2}{2\sigma^2}\right] \exp\left[-\frac{(y_s - \bar{y}_s)^2}{2\sigma_s^2}\right]. \tag{34}$$

It should be noted that no normalization constant is required in the present case because λ_c is defined for all real value of y_s. It follows immediately that the reliability is

$$R = \int_{0+}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi\sigma\,\sigma_s} \exp\left[-\frac{(y_s - \bar{y}_s)^2}{2\sigma_s^2}\right] \exp\left[\frac{(\lambda_R - 1 + y_s^{2/3} + \bar{y})^2}{2\sigma^2}\right] dy_s\, d\lambda_R. \tag{35}$$

Integration with respect to λ_R yields the result

$$R = \frac{1}{2} - \int_{-\infty}^{\infty} \frac{1}{2\sigma_s\sqrt{2\pi}} \exp\left[-\frac{(y_s - \bar{y}_s)^2}{2\sigma_s^2}\right] \operatorname{erf}\left(\frac{-1 + y_s^{2/3} + \bar{\lambda}}{\sigma\sqrt{2}}\right) dy_{s'}. \tag{36}$$

This result is valid for all systems in which the imperfection sensitivity reduces to the form given in equation (22). Some results are shown in Figs. 4–6.

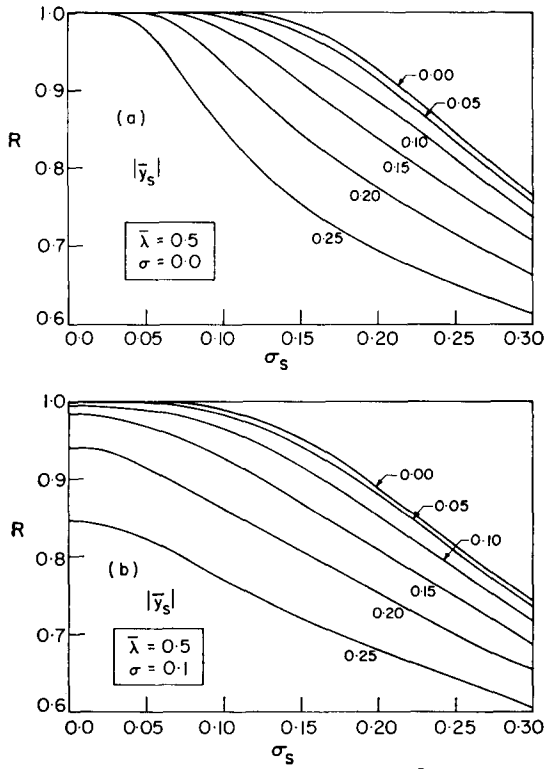

Fig. 4. Symmetric system: Influence of the inputs $\bar{\lambda}$, σ, $\bar{y}_s$ and σ_s on R.

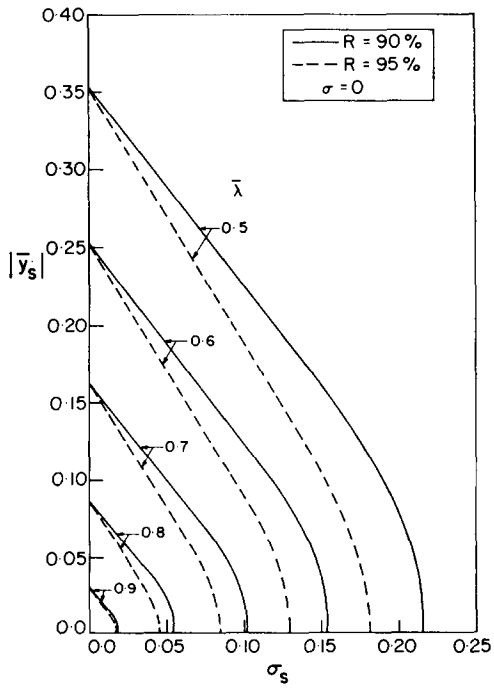

Fig. 5. Symmetric system: Equal levels of R in the $|\bar{y}_s| - \sigma_s$ plane as $\bar{\lambda}$ varies.

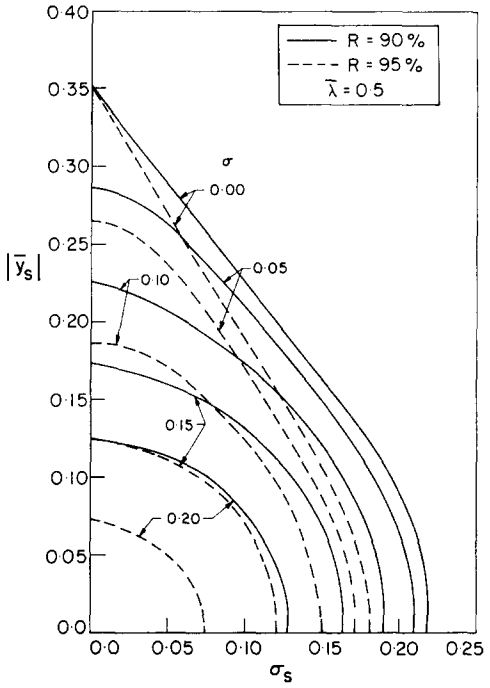

Fig. 6. Symmetric system: Equal levels of R in the $|\bar{y}_s| - \sigma_s$ plane as σ varies.

DISCUSSION OF RESULTS

Some interesting features appear in the results shown in Figs. 2 and 5. The first point of interest lies in the convergence to the same point on the vertical axis of curves for different reliabilities but the same deterministic applied load. The second point of interest is the complete linearity of the curves in Fig. 2 and the partial linearity of curves in Fig. 5 (for large values of $\bar{y}_s/\sigma_s$).

A further reduction of equations (31) and (36) will shed some light on these features. An evaluation of R for $\sigma = 0$, using equation (30) and substitution of R into equation (31) yields, after some algebra, the following simplified expression for R_T.

$$R_T = \tfrac{1}{2}\left[1 + \operatorname{erf}\left(\frac{(1-\bar{\lambda})^2 - \bar{y}_A}{\sqrt{2}\,\sigma_A}\right)\right], \qquad 0 \le \bar{\lambda} \le 1. \tag{37}$$

This equation can be transformed to

$$\bar{y}_A = (1-\bar{\lambda})^2 - \sqrt{2}\,[\operatorname{erf}^{-1}(2R_T - 1)]\sigma_A \tag{38}$$

which demonstrates the linear correspondence between $\bar{y}_A$ and σ_A.

A similar reduction of equation (36) yields

$$R = \tfrac{1}{2}\operatorname{erf}\left(\frac{(1-\bar{\lambda})^{3/2} + \bar{y}_s}{\sqrt{2}\,\sigma_s}\right) + \tfrac{1}{2}\operatorname{erf}\left(\frac{(1-\bar{\lambda})^{3/2} - \bar{y}_s}{\sqrt{2}\,\sigma_s}\right), \qquad 0 \le \bar{\lambda} \le 1, \tag{39}$$

for the symmetric case. If the ratio $\bar{y}_s/\sigma_s$ is large (say > 3) equation (39) can be transformed to the form

$$\bar{y}_s = (1 - \bar{\lambda})^{3/2} - \sqrt{2}\,[\mathrm{erf}^{-1}(2R - 1)]\sigma_s, \tag{40a}$$

or if $-\bar{y}_s/\sigma_s$ is large (>3) to the form

$$\bar{y}_s = -(1 - \bar{\lambda})^{3/2} + \sqrt{2}\,[\mathrm{erf}^{-1}(2R - 1)]\sigma_s. \tag{40b}$$

Equations (38) and (40) show that the slope of the corresponding curves in Figs. 2 and 5 depend only on the value of reliability and that the slope in the asymmetric case is equal to that in the symmetric case (for large values of $\bar{y}_s/\sigma_s$) for a given value of reliability. In both cases the intercept with the vertical axis ($\sigma_A = 0$ or $\sigma_s = 0$) is independent of the reliability. This is of course to be expected since the whole system reverts back to being completely deterministic.

The above results also embrace the results of [3] as a special case. The equivalence arises if the initial imperfection is assumed to be a deterministic shape with a random amplitude. A stability definition of the form of equation (6) is also inherent in the development of the results in [3].

EXAMPLE: BEAM ON ELASTIC FOUNDATION

Consider the problem of a beam resting on a non-linear elastic foundation a shown in Fig. 7. The model is a deterministic system which exhibits idealized behaviour. This model, subjected to load L, is used to determine the ideal buckling load and buckling shape. The restricted system is a random system and its geometry represents the real geometry. The load applied to the restricted system is the random critical load. This load is chosen such that almost every sample is in an unstable critical state of equilibrium. The real system is the random system under consideration. In Fig. 7(c) $w(x)$ represents the deflection, $w_0(x)$ the initial imperfection and $\bar{L}$ the random applied load.

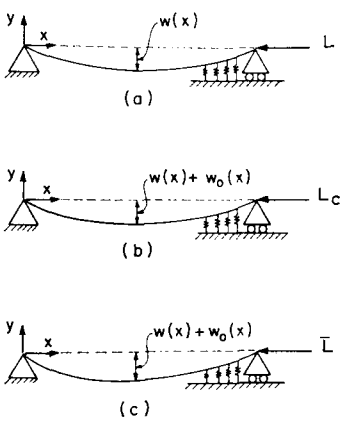

Fig. 7. Beam on nonlinear elastic foundation. (a) The model. (b) The restricted system. (c) The real system.

(1) *The potential energy of the model*

The total potential energy for the model is given by:

$$V - W = \tfrac{1}{2} EI\int_0^l \left\{ \frac{\left(\frac{d^2w}{dx^2}\right)^2}{1-\left(\frac{dw}{dx}\right)^2} \right\} dx + L\int\left\{\left[1-\left(\frac{dw}{dx}\right)^2\right]^{1/2} - 1\right\} dx + \int_0^l \left[\int_0^w \bar{q}(w)\, dw\right] dx. \tag{41}$$

In the above, V represents the strain energy of the system, W the work done by the applied load L, E the Young's modulus, I the moment of inertia of a cross-section of the beam, l the length of the beam, and $\bar{q}(w)$ the force per unit deflection of the elastic foundation. In this expression it is assumed that the beam is axially rigid. Thus the first term represents the strain energy due to bending, the second represents the work done by the applied load and the third the strain energy absorbed by the elastic foundation. Consider now the non-dimensionalization:

$$\xi = \frac{x}{l},\ \Psi = \frac{w}{l},\ \lambda = \frac{Ll^2}{EI},\ q(\Psi) = \frac{\bar{q}(w)}{EI} l^2, \quad \text{and} \quad P^\lambda = \frac{2l}{EI}(V - W). \tag{42}$$

Then equation (41) becomes

$$P^\lambda[\Psi] = \int_0^1 \frac{(\Psi'')^2}{1-(\Psi')^2}\, d\xi + 2\lambda\int_0^1 \{[1-(\Psi')^2]^{1/2} - 1\}\, d\xi + 2\int_0^1 \left[\int_0^\Psi q(\Psi)\, d\Psi\right] d\xi, \tag{43}$$

where

$$\Psi' = d\Psi/d\xi \qquad \text{and} \qquad q(\Psi) = K_1\Psi + K_2\Psi^2 + K_3\Psi^3.$$

The parameters K_1, K_2 and K_3 are the linear, quadratic and cubic foundation constants, respectively.

Expanding the integrand in series, and retaining only those terms up to and including the fourth degree in Ψ and its derivatives, yields

$$\begin{aligned} P_1^\lambda[\Psi] &= 0, \\ P_2^\lambda[\Psi] &= \int_0^1 [(\Psi'')^2 - \lambda(\Psi')^2 + K_1\Psi^2]\, d\xi, \\ P_3^\lambda[\Psi] &= \int_0^1 2K_2\frac{\Psi^3}{3}\, d\xi, \\ P_4^\lambda[\Psi] &= \int_0^1 \left[(\Psi''\Psi')^2 + K_3\frac{\Psi^4}{2} - \frac{\lambda(\Psi')^4}{4}\right] d\xi, \end{aligned} \tag{44}$$

where $P_i^\lambda[\Psi]$ represents that term which is of the ith degree in Ψ and its derivatives in the potential energy.

(2) *The buckling load of the model*

The buckling load of the model is given by the smallest eigenvalue of the linearized problem. This is obtained by setting the first variation of the quadratic terms, $P_2^\lambda[\Psi]$, to zero. Doing so, yields

$$\int_0^1 [\Psi''\,\delta\Psi'' - \lambda\Psi'\,\delta\Psi' + K_1\Psi\,\delta\Psi]\, d\xi = 0. \tag{45}$$

Integration by parts and making use of the arbitrariness of $\delta\Psi'$ yields

$$\Psi'' = 0 \qquad \text{at} \qquad \xi = 0, 1, \tag{46}$$

and the equilibrium equation

$$\Psi^{IV} + \lambda\Psi'' + K_1\Psi = 0. \tag{47}$$

The forced boundary conditions are

$$\Psi = 0 \qquad \text{at} \qquad \xi = 0, 1. \tag{48}$$

Equation (47) and the corresponding boundary conditions yield an eigenvalue problem, the eigenfunction of which is

$$\Psi_1 = \sin m\pi\xi, \tag{49}$$

and the eigenvalue

$$\lambda_1 = \frac{(m\pi)^4 + K_1}{(m\pi)^2}. \tag{50}$$

The value of m is such that λ_1 is a minimum and m is determined from the inequality [12],

$$m^2(m+1)^2\pi^4 > K_1 > (m-1)^2m^2\pi^4. \tag{51}$$

(3) *Stability of the buckling state*

The stability of the buckling state ($\lambda = \lambda_1$, $\Psi = \Psi_1$) is determined by the character of third and higher order terms in the potential energy expansion. Thus

$$A_3 = P_3(\Psi_1) = \int_0^1 \tfrac{2}{3}K_2\Psi_1{}^3\,\mathrm{d}\xi = \frac{8K_2}{9m\pi}\frac{1}{2}(1-(-1)^m). \tag{52}$$

Therefore if $A_3 \neq 0$ the system is unstable in the bifurcation state and, as a result, is imperfection sensitive. This system would be designated as an asymmetric system. If $A_3 = 0$, it is necessary to investigate the fourth order terms. Thus

$$A_4 = P_4(\Psi_1) = \int_0^1 [(\Psi_1'\Psi_1'')^2 + \tfrac{1}{2}K_3\Psi_1{}^4 - \tfrac{1}{4}\lambda_1(\Psi_1')^4]\,\mathrm{d}\xi = \frac{(m\pi)^6}{32} - \frac{3(m\pi)^2}{32}K_1 + \frac{3K_3}{16}. \tag{53}$$

In equations (52) and (53), $P_n(\Psi)$ represents the coefficient of the first term in the Taylor series expansion of $P_n{}^{\lambda}(\Psi)$ about the point $\lambda = \lambda_1$.

(4) *The restricted system*

Consider now the random potential energy expression of the restricted system. For this system, an expression similar to that of the model (equation (44)) is obtained, the difference being due to the additional presence of terms dependent on the initial imperfection, and a replacement of L by L_c. Using the previous dimensionless quantities with the new ones

$$\Psi_0 = \frac{w_0}{l}, \qquad \lambda_c = \frac{L_c l^2}{EI}, \tag{54}$$

yields the restricted potential energy functional

$$P^{\lambda_c}[\Psi] + Q^{\lambda_c}[\Psi] = \int_0^1 \left[\frac{\Psi'' + \Psi_0''}{\sqrt{-1(\Psi' + \Psi_0')^2}} - \frac{\Psi_0''}{\sqrt{(1-(\Psi_0')^2}}\right]^2 d\xi + 2\lambda_c \int_0^1 \{[1-(\Psi' + \Psi_0')^2]^{1/2} - [1-(\Psi_0')^2]^{1/2}\} d\xi + 2\int_0^1 \left\{\int_0^{\Psi} [q(\Psi + \Psi_0) - q(\Psi_0)]\, d\Psi\right\} d\xi, \quad (55)$$

where $p^{\lambda_c}[\Psi]$ represents those terms independent of Ψ_0, and $Q^{\lambda_c}[\Psi]$ represents those terms containing Ψ_0.

Expanding the potential energy expression in series and retaining only the predominant terms yields

$$\begin{aligned} P_1{}^{\lambda_c}[\Psi] &= 0, \\ P_2{}^{\lambda_c}[\Psi] &= \int_0^1 [(\Psi'')^2 - \lambda_c(\Psi')^2 + K_1\Psi^2]\, d\xi, \\ P_3{}^{\lambda_c}[\Psi] &= \int_0^1 [\tfrac{2}{3}K_2\Psi^3]\, d\xi, \\ P_4{}^{\lambda_c}[\Psi] &= \int_0^1 [(\Psi''\Psi')^2 + \tfrac{1}{2}K_3\Psi^4 - \tfrac{1}{4}\lambda_c(\Psi')^4]\, d\xi, \\ Q_1{}^{\lambda_c}[\Psi] &= -2\lambda_c \int_0^1 [\Psi_0'\Psi']\, d\xi, \end{aligned} \quad (56)$$

where $P_i{}^{\lambda_c}[\Psi]$ is as previously defined and $Q_1{}^{\lambda_c}[\Psi]$ represents those terms linear in Ψ_0 and of 1st degree in Ψ.

Further expansion of the potential energy, this time with respect to the load parameter, about $\lambda_c = \lambda_1$ yields

$$P^{\lambda_c}[\Psi] + Q_1{}^{\lambda_c}[\Psi] = \int_0^1 [(\Psi'')^2 - \lambda_1(\Psi')^2 + K_1\Psi^2]\, d\xi + \int_0^1 [\tfrac{2}{3}K_2\Psi^3]\, d\xi + \int_0^1 [(\Psi''\Psi')^2 + \tfrac{1}{2}K_3\Psi^4 - \tfrac{1}{4}\lambda_1(\Psi')^4]\, d\xi - 2\lambda_1 \int_0^1 [\Psi_0'\Psi']\, d\xi + \int_0^1 (\lambda_1 - \lambda_c)(\Psi')^2\, d\xi. \quad (57)$$

It should be noted that in equation (57) only the predominant term in $(\lambda_1 - \lambda_c)$ has been retained.

(5) *The asymmetric case* ($K_2 \neq 0$ and m is odd)

In the asymmetric case the important system constants in equation (11) are

$$\begin{aligned} B_1 &= Q_1[\Psi_1] = -2(m\pi)\lambda_1 \int_0^1 \Psi_0' \cos m\pi\xi\, d\xi, \\ A_2' &= P_2'[\Psi_1] = -\int_0^1 (\Psi_1')^2\, d\xi = -\frac{(m\pi)^2}{2}, \\ A_3 &= P_3[\Psi_1] = \int_0^1 \tfrac{2}{3}K_2\Psi_1{}^3\, d\xi = \frac{8K_2}{9(m\pi)}. \end{aligned} \quad (58)$$

In the above $Q_1[\Psi_1]$ is the coefficient of the first term in the Taylor expansion of $Q_1{}^{\lambda_c}[\Psi]$ about $\lambda_c = \lambda_1$, $P_2'[\Psi_1]$ is the coefficient of the second term in the expansion of $P_2{}^{\lambda_c}[\Psi]$ about $\lambda_c = \lambda_1$ and so on. B_1 can be simplified through integration by parts. Using the

condition that $\Psi_0 = 0$ at 0 and 1 in order that the initial imperfection be consistent with the boundary conditions, one obtains

$$B_1 = -2(m\pi)^2\lambda_1 \int_0^1 \Psi_0 \sin m\pi\xi \, d\xi. \tag{59}$$

The density function of the imperfection parameter requires the definition of the mean $\bar{y}_A$ and the standard deviation σ_A. These quantities are defined except for the expectations $E[B_1]$ and $E[B_1{}^2]$. Taking the expected value of equation (59) yields

$$E[B_1] = -2(m\pi)^2\lambda_1 \int_0^1 E[\Psi_0]\sin m\pi\xi \, d\xi. \tag{60}$$

It is readily seen that

$$b(m) = 2\int_0^1 E[\Psi_0]\sin m\pi\xi \, d\xi, \tag{61}$$

is the mth Fourier sine coefficient of $E[\Psi_0]$.
Therefore

$$E[B_1] = -(m\pi)^2\lambda_1 b(m). \tag{62}$$

Squaring both sides of equation (59) and then taking the expected value of the result yields

$$\begin{aligned} E[B_1{}^2] &= 4(m\pi)^4\lambda_1{}^2 E\left\{\left[\int_0^1 \Psi_0 \sin m\pi\xi \, d\xi\right]^2\right\}, \\ &= 4(m\pi)^4\lambda_1{}^2 E\left\{\left[\int_0^1 \Psi_0(\xi_1)\sin m\pi\xi_1 \, d\xi_1\right]\left[\int_0^1 \Psi_0(\xi_2)\sin m\pi\xi_2 \, d\xi_2\right]\right\}, \\ &= 4(m\pi)^4\lambda_1{}^2\int_0^1\int_0^1 E[\Psi_0(\xi_1)\Psi_0(\xi_2)]\sin m\pi\xi_1 \sin m\pi\xi_2 \, d\xi_1 \, d\xi_2. \end{aligned} \tag{63}$$

The term $E[\Psi_0(\xi_1)\Psi_0(\xi_2)]$ represents the autocorrelation of Ψ_0. Also

$$S(m, m) = 4\int_0^1\int_0^1 E[\Psi_0(\xi_1)\Psi_0(\xi_2)]\sin m\pi\xi_1 \sin m\pi\xi_2 \, d\xi_1 \, d\xi_2, \tag{64}$$

is the coefficient of the (m, m)th term of the fourier sine series of the autocorrelation and can thus be considered as the generalized spectral density of Ψ_0 evaluated at the frequencies which correspond to the (m, m)th modes. Therefore

$$E[B_1{}^2] = (m\pi)^4\lambda_1{}^2 S(m, m). \tag{65}$$

Using the above, the expected value and standard deviation of the imperfection parameter are, respectively,

$$\bar{y}_A = -\frac{32K_2 b(m)}{(3m\pi)^5 + 3(m\pi)K_1}, \tag{66}$$

$$\sigma_A = \frac{32|K_2|}{(3m\pi)^5 + 3(m\pi)K_1}[S(m, m) - b^2(m)]^{1/2}. \tag{67}$$

Referring now to Figs. 1–3 the influence of the various parameters on the reliability of the system can be determined.

(6) *The symmetric case ($K_2 = 0$ and/or m is even)*

In the symmetric case the system constants are similar to those of equation (58) except that A_3 vanishes and

$$A_4 = P_4[\Psi_1] = \frac{(m\pi)^6}{32} - \frac{3(m\pi)^2 K_1}{32} + \frac{3K_3}{16}. \tag{68}$$

It immediately follows from equations (32) and (33), respectively, that

$$\bar{y}_s = -\left\{\frac{27[(m\pi)^6 - 3(m\pi)^2 K_1 + 6K_3]}{-32[(m\pi)^4 + K_1]}\right\}^{1/2} b(m), \tag{69}$$

and

$$\sigma_s = \left\{\frac{27[(m\pi)^6 - 3(m\pi)^2 K_1 + 6K_3]}{-32[(m\pi)^4 + K_1]}\right\}^{1/2} [S(m, m) - b^2(m)]^{1/2}. \tag{70}$$

As in the asymmetric case these quantities can be superimposed on the appropriate axes in Figs. 4–6 to determine the influence of varying the different parameters.

SUMMARY AND CONCLUSIONS

It is shown in this paper that useful probabilistic stability criteria can be formulated on the basis of two fundamental concepts. These are; (i) almost sure sample stability, and (ii) sample stability in probability (i.e. reliability). The latter concept is used to study the stability problem of imperfection sensitive systems in which both the initial imperfections and the applied loads are random quantities. Use of Koiter's method and restriction of the analysis to only the first order approximation yields certain general results that are applicable to a wide class of elastic systems. These results may be summarized as follows:

(1) The stability boundary, and the stability coefficient of an imperfect structural system are independent of the initial imperfection. They are inherent characteristics of the perfect model (or ideal system) and can be obtained from a deterministic analysis. A deterministic study of the model should always precede a probabilistic study of the real structure.

(2) If the applied load is assumed to be a Gaussian random variable, and the initial imperfection a Gaussian process, and if these two variables are independent, then the reliability of the imperfect system can be found in terms of the mean and autocorrelation of the geometric imperfections and the mean and standard deviation of the applied load.

(3) Both the mean and the autocorrelation of the initial imperfections of the structure play an important role in a probabilistic stability analysis. For the example of a randomly imperfect beam and a nonlinear elastic foundation some of these ideas can be further developed.

(4) The Fourier sine coefficient of the mean initial imperfection that corresponds to the buckling mode of the structure is of prime importance. It bears a linear relationship to the mean imperfection parameter which is used in the analysis.

(5) The generalized spectral density of the initial imperfections, evaluated at the buckling mode frequencies, is of equal importance. It is linearly related to the mean square of the imperfection parameter.

The stability criteria and the analysis presented in this paper can also be used in elastic plate and shell stability problems, provided a suitable joint density function of the type appearing in equation (18) or (25) can be found.

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Резюме — Понятие о безошибочной стабильности образца и возможной стабильности образца формулируется для эластичных систем. Применяя аппроксимацию Койтера эти понятия используются при анализе дефектных чувствительных структур. Приложенную нагрузку и начальные геометрические дефекты вводятся в анализ как произвольные величины. Сжатая балка финитной длины на нелинейном эластичном фундаменте применяется в примерном вычислении.